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Algebraic and geometric lifting

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Abstract. Every spread in $PG(3,q)$ may be constructed from a mixed subgeometry partition of $PG(3,q^2)$.

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1 Geometric and algebraic lifting

In the text by Hirschfeld and Thas [3], there is given a construction of finite spreads, and hence, finite translation planes, from either Baer subgeometry partitions or mixed subgeometry partitions of a finite projective space. The partitions are the points of finite projective geometries Σ over $GF(q^2)$. When Σ is isomorphic to $PG(2m,q^2)$, the partition components are Baer subgeometries isomorphic to $PG(2m,q)$. When Σ is isomorphic to $PG(2n-1,q^2)$, $n>1$, it is possible to have a so-called 'mixed' partition of β $PG(n-1,q^2)$'s and α $PG(2n-1,q)$'s. The configuration is such that $\alpha(q+1)+\beta = q^{2n}+1$.

The interest in such partitions lies in the fact they may be used to construct spreads and hence translation planes. Baer subgeometry partitions produce translation planes of order q^{2m+1} with kernel containing $GF(q)$, whereas mixed partitions produce translation planes of order q^{2n} and kernel containing $GF(q)$. The process is called 'geometric lifting' in Johnson [5]. Hirschfeld and Thas constructed some mixed partitions of $PG(3,q^2)$, which turn out to construct André spreads of order q^4. The main unanswered question is what spreads may be constructed using geometric lifting. For example, there does not seem to be a direct connection between spreads in $PG(3,q)$ and translation planes of order q^2 and subgeometry partitions of $PG(3,q^2)$ since partitions produce translation planes of order q^4. Or rather, to directly construct a translation plane of order q^2 with spread in $PG(3,q)$ from a subgeometry partition, we would need that $n = 1$, which is prohibited in the geometric lifting construction, but see below for an attempt at such constructions when n is indeed chosen to be 1.

In Johnson [5], the question was considered as to how to recognize spreads that have been geometrically lifted from Baer subgeometry or mixed subgeometry partitions of a finite projective space. It was found that the intrinsic character is that the translation plane have order q^t with subkernel K isomorphic to $\mathrm{GF}(q)$ and admit a fixed-point-free collineation group (on the nonzero vectors) which contains the scalar group K^*, written as GK^*, such that GK^* union the zero mapping is a field isomorphic to $\mathrm{GF}(q^2)$ (see Johnson [5] and Johnson-Mellinger [6]). With such a recognition theorem on collineation groups, it is then possible to 'retract' such a translation plane or spread to construct a variety of Baer subgeometry or mixed subgeometry partitions of an associated projective space written over GK as a quadratic field extension of K. To construct spreads in $\mathrm{PG}(3,q)$ directly requires subgeometry partitions of $\mathrm{PG}(1,q^2)$, which of course are simply lines ($\mathrm{PG}(1,q)$'s). In this case, there will be a collineation group in the associated translation plane of order q^2-1, which fixes all components; that is, the translation plane of order q^2 has $\mathrm{GF}(q^2)$ as a kernel homology group, so only the Desarguesian plane may be so constructed.

Hence, it would seem that the only spreads that can be obtained from a mixed subgeometry partition of $\mathrm{PG}(3,q^2)$ are those spreads corresponding to translation planes of order q^4 that admit the required 'field group' of order q^2-1. However, there is also an algebraic construction procedure for spreads, which is called 'algebraic lifting' (or more simply 'lifting' in Johnson [4]) by which a spread in $\mathrm{PG}(3,q)$ may be lifted to a spread in $\mathrm{PG}(3,q^2)$. More precisely, this construction is a construction on the associated quasifields for the spread and different quasifields may produce different algebraically lifted spreads. The reverse procedure of constructing spreads in $\mathrm{PG}(3,q)$ from certain spreads in $\mathrm{PG}(3,q^2)$ is called 'algebraic contraction'. This material is explicated in Biliotti, Jha and Johnson and the reader is referred to this text for additional details and information (see [1]).

Now apart from the name, there should be no connection between geometric lifting from a subgeometry partition of a projective space to a spread and algebraic lifting from a subspace partition of a vector space to a spread. On the other hand, when a spread in $\mathrm{PG}(3,q)$ is algebraically lifted to a spread in $\mathrm{PG}(3,q^2)$, it turns out that there is a suitable field group of order q^2-1, from which such an algebraically lifted spread of order q^4 can be produced from a mixed subgeometry partition. This spread in $\mathrm{PG}(3,q^2)$ begins its existence as a spread in $\mathrm{PG}(7,q)$, but actually has kernel isomorphic to $\mathrm{GF}(q^2)$. This means that our original spread in $\mathrm{PG}(3,q)$ may be constructed by a 2-step procedure from a mixed subgeometry partition of $\mathrm{PG}(3,q^2)$. Previously, in Johnson and

Mellinger [6], this idea of producing a spread from a series of construction methods is considered, where it is pointed out that one may algebraically lift a spread in $\mathrm{PG}(3,q)$ to a spread in $\mathrm{PG}(3,q^2)$, which is actually derivable. The derived translation plane produces a spread in $\mathrm{PG}(7,q)$, and the associated plane admits a field group (the original kernel group of the spread in $\mathrm{PG}(3,q^2)$), which may be used to produce a mixed subgeometry partition of $\mathrm{PG}(3,q^2)$. Then the construction procedure to obtain the original spread requires a 3-step procedure: geometric lifting—derivation—algebraic contraction.

In this note, we establish the following fundamental connection:

1 Theorem. *Let S be any spread in $\mathrm{PG}(3,q)$. Then there is a mixed subgeometry partition of $\mathrm{PG}(3,q^2)$, which geometrically lifts to a spread in $\mathrm{PG}(3,q^2)$ that algebraically contracts to S.*

2 Corollary. *The set of mixed subgeometry partitions of a 3-dimensional projective space $\mathrm{PG}(3,k^2)$, constructs all spreads of $\mathrm{PG}(3,k)$.*

2 Background on algebraic lifting and spread retraction

In this section, we give background on the construction procedure of algebraic lifting. Part of this material is a variation of material from from Biliotti, Jha and Johnson [1].

3 Definition (Lifting)**.** Let $K = \mathrm{GF}(q)$ be any finite field and choose $\overline{K} = K(\theta) \simeq \mathrm{GF}(q^2)$, defined by some $\theta \in \overline{K} \setminus K$ with minimal polynomial $\theta^2 = \theta\alpha + \beta$, with $\alpha \in K$, $\beta \in K^*$. Let σ denote the involutory automorphism in $\mathrm{Gal}(\overline{K}/K)$, so $\sigma\colon \overline{K} \to \overline{K}$ may be expressed as: $\sigma\colon \theta t + u \mapsto -\theta t + u + \alpha t\ \forall\, u, t \in K$.

Suppose (g, f) defines a spread set $\mathcal{M}_{(g,f)}$ on K, and define $h(t, u)$ by $f(t, u) = h(t, u) - \alpha g(t, u)$. Hence, the spread set is

$$\begin{bmatrix} g(t,u) & h(t,u) - \alpha g(t,u) \\ t & u \end{bmatrix} \ \forall\, u, t \in K.$$

Then the θ-**lifting** of the spread set $\mathcal{M}_{(g,f)}$ is the following set of 2×2 matrices on $\overline{K}$:

$$\mathcal{M}_{(g,f)}(\theta) := \left\{ \begin{bmatrix} v^q & H(u) \\ u & v \end{bmatrix} \middle|\, u, v \in \overline{K} \right\},$$

where the function $H\colon \overline{K} \to \overline{K}$ is defined by

$$H(\theta t + u) = -g(t,u)\theta + h(t,u) := -g(t,u)\theta + (f(t,u) + \alpha g(t,u)),\ \forall\, u, t \in K.$$

The following theorem asserts that spread sets over $\mathrm{GF}(q)$ lift to σ-associated spread sets over $\mathrm{GF}(q^2)$. Hence, any line spread in $\mathrm{PG}(3,q)$ 'algebraically lifts' to a line spread in $\mathrm{PG}(3,q^2)$. The theorem implies a characterization of lifted line spreads.

4 Theorem (Lifting spread sets). *Let $\overline{K} = \mathrm{GF}(q^2) \supset \mathrm{GF}(q) = K$, σ the involution in $\mathrm{Gal}(\overline{K})$, and choose $\theta \in \overline{K} \setminus K$ so $\theta^2 = \theta\alpha + \beta$ for $(\alpha, \beta) \in K^* \times K$. Using the notation of Definition 3, let*

$$H(\theta t + u) := -g(t,u)\theta + h(t,u) := -g(t,u)\theta + (f(t,u) + \alpha g(t,u)),\ \forall\, u, t \in K.$$

*Then the θ-**lifting** of the spread set $\mathcal{M}_{(g,f)}$,*

$$\mathcal{M}_{(g,f)}(\theta) = \left\{ \begin{bmatrix} v^q & H(u) \\ u & v \end{bmatrix} \,\middle|\, u, v \in \overline{K} \right\}, \tag{1}$$

is a spread set.

Conversely, assume that the collection of matrices $\mathcal{M}_{(g,f)}(\theta)$ defined by (1) turns out to be a spread set. Then the collection of K-matrices $\mathcal{M}_{(g,f)}$, defined as indicated in Definition 3, is also a spread set (and hence lifts to the spread set $\mathcal{M}_{(g,f)}(\theta)$) and is an 'algebraic contraction' of the spread set $\mathcal{M}_{(g,f)}(\theta)$.

5 Definition. We call any derivable net isomorphic to

$$x = 0, y = x \begin{bmatrix} u^q & 0 \\ 0 & u \end{bmatrix}; u \in \mathrm{GF}(q^2),$$

a 'σ-invariant net'.

6 Corollary (Lifted finite spreads). *Let σ be the involutory automorphism of $\mathrm{GF}(q^2)$, and suppose π is a line spread in $\mathrm{PG}(3,q^2)$.*

*Then π is lifted from a line spread in $\mathrm{PG}(3,q)$ if and only if π admits an elation group E of order q^2 whose non-trivial component orbits define σ-derivable E-nets and π admits a non-trivial K-linear Baer group B of order > 2 that normalizes but does **not** centralize E.*

When π is of this form, the full Baer group B fixing π_B contains a Baer subgroup of order $q+1$ that normalizes but does not centralize E.

We will be using the following theorem of Hiramine, Matsumoto and Oyama [2] (using our terminology).

7 Theorem. *Let π be a translation plane of order q^4 with spread in $\mathrm{PG}(3,q^2)$. If π admits a Baer group B of order $q+1$ and an elation group E of order at least q^2 such that $[E,B] \neq 1$ then π is an algebraically lifted plane.*

Let an algebraically lifted plane have spread

$$x=0, y=x\begin{bmatrix} u & F(t) \\ t & u^q \end{bmatrix}; t,u\in \mathrm{GF}(q^2),\ F\colon \mathrm{GF}(q^2)\to \mathrm{GF}(q^2),\ F(0)=0.$$

Then as noted above, every such plane admits an elation group E of order q^2

$$E=\left\langle \begin{bmatrix} 1 & 0 & u & 0 \\ 0 & 1 & 0 & u^q \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; u\in \mathrm{GF}(q^2)\right\rangle,$$

and a Baer group B of order $q+1$

$$\left\langle \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e \end{bmatrix}; e^{q+1}=1,\ e\in \mathrm{GF}(q^2)^*\right\rangle,$$

and of course, the kernel homology group K^* of order $q-1$

$$\left\langle \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}; a\in \mathrm{GF}(q^2)-\{0\}\right\rangle.$$

The Baer group fixes exactly q^2+1 components of the Baer subplane

$$\pi_0=\big\langle (0,x_2,0,y_2); x_2,y_2\in \mathrm{GF}(q^2)\big\rangle$$

that is pointwise fixed by B.

2.1 Spread retraction

We shall require the following theorems, which are due to Johnson [5].

8 Theorem. *Let π be a translation plane with spread in $\mathrm{PG}(4m-1,q)$. Suppose the associated vector space may be written over a field K isomorphic to $\mathrm{GF}(q^2)$ which extends the indicated field $\mathrm{GF}(q)$ as a $2m$-dimensional K-vector space.*

If the scalar mappings with respect to K over V_{2m}/K act as collineations of π, assume that the orbit lengths of components are either 1 or $q+1$ under the scalar group of order q^2-1.

Let δ denote the number of components of orbit length 1 and let $(q+1)d$ denote the number of components of orbit length $q+1$.

Then there is a mixed partition of $\mathrm{PG}(2m-1,q^2)$ of δ $\mathrm{PG}(m-1,q^2)$'s and d $\mathrm{PG}(2m-1,q)$'s.

9 Definition. Under the above conditions, we shall say that the mixed partition of $\mathrm{PG}(2m-1, q^2)$ is a 'retraction' of the spread of π or a 'spread retraction'.

10 Theorem. *Let π be a translation plane of order q^{2m+1} with kernel containing $\mathrm{GF}(q)$, with spread in $\mathrm{PG}(4m+1, q)$, whose underlying vector space is a $\mathrm{GF}(q^2)$-space and which admits as a collineation group the scalar group of order q^2-1. If all orbits of components have length $q+1$ corresponding to $K-\{0\}$, then a Baer subgeometry partition of $\mathrm{PG}(2m, q^2)$ may be constructed.*

11 Definition. A Baer subgeometry partition produced from a spread as above is called a 'spread retraction'.

In this article, we shall be interested in mixed subgeometry partitions of $\mathrm{PG}(3, q^2)$, each of which is then a set of projective subspaces isomorphic to $\mathrm{PG}(1, q^2)$'s and a set of $\mathrm{PG}(3, q)$'s that partition the projective space. From the retraction theorem above, we would need a spread in $\mathrm{PG}(7, q)$ with an appropriate collineation group to accomplish this. The basic idea is that from a spread in $\mathrm{PG}(3, q)$, there is an algebraically lifted spread in $\mathrm{PG}(3, q^2)$, which may also be regarded as a spread in $\mathrm{PG}(7, q)$. So, we are able to locate a suitable spread if we can find a suitable collineation group.

3 Subgeometry partitions and spreads in $\mathrm{PG}(3, q)$

Apart from background on algebraic lifting, we also used the retraction theorem of Johnson.

In this section, we give the proof to the fundamental theorem:

12 Theorem. *Let S be any spread in $\mathrm{PG}(3, q)$. Then there is a mixed subgeometry partition of $\mathrm{PG}(3, q^2)$, which geometrically lifts to a spread in $\mathrm{PG}(3, q^2)$ that algebraically contracts to S.*

13 Corollary. *The set of mixed subgeometry partitions of a 3-dimensional projective space $\mathrm{PG}(3, k^2)$, constructs all spreads of $\mathrm{PG}(3, k)$.*

Proof. Let π be any translation plane with spread in $\mathrm{PG}(3, q)$. Consider any matrix spread set $\mathcal{M}$, and form the algebraic lift to a spread set in $\mathrm{PG}(3, q^2)$. This spread admits a Baer group B of order $q+1$. Let $K^*_{q^2-1}$ denote the kernel homology group of order q^2-1 of the lifted plane. We note that any lifted spread has the general form

$$x = 0, y = x \begin{bmatrix} u & F(t) \\ t & u^q \end{bmatrix}; u, t \in \mathrm{GF}(q^2),$$

where $F\colon \mathrm{GF}(q^2) \to \mathrm{GF}(q^2)$, such that $F(0) = 0$. The Baer group has the following form:

$$\left\langle \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e \end{bmatrix} ; e^{q+1} = 1,\ e \in \mathrm{GF}(q^2)^* \right\rangle .$$

Write B as follows:

$$B = \left\langle \tau_a = \begin{bmatrix} a^{q-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a^{q-1} \end{bmatrix} ; a \in \mathrm{GF}(q^2)^* \right\rangle .$$

Note that the kernel homology group $K^*_{q^2-1}$ may be written in the form

$$\left\langle k_a = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix} ; a \in \mathrm{GF}(q^2)^* \right\rangle .$$

Now form the group with elements $\tau_a k_a$:

$$\left\langle \tau_a k_a = \begin{bmatrix} a^q & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^q \end{bmatrix} ; a \in \mathrm{GF}(q^2)^* \right\rangle .$$

As a matrix ring,

$$\mathcal{K} = \left\langle \tau_a k_a = \begin{bmatrix} a^q & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^q \end{bmatrix} ; a \in \mathrm{GF}(q^2) \right\rangle$$

is clearly a field isomorphic to $\mathrm{GF}(q^2)$, whose multiplicative group acts fixed-point-free. Note that when $a \in \mathrm{GF}(q)^*$, we obtain a kernel subgroup of order $q-1$. Hence, the associated lifted spread produces a mixed subgeometry partition. Applying the reverse construction process to algebraic lifting (algebraic contraction), we have the proof to the theorem. $\boxed{QED}$

14 Remark. The component orbits under $\mathcal{K}^*$ are the orbits under B, hence there are exactly q^2+1 components fixed by $\mathcal{K}^*$ and $(q^4-q^2)/(q+1) = q^2(q-1)$, $\mathrm{PG}(3,q)$'s. Hence, there is a mixed subgeometry partition in $\mathrm{PG}(3,q^2)$ of q^2+1 $\mathrm{PG}(1,q^2)$'s and $q^2(q-1)$ $\mathrm{PG}(3,q)$'s. Therefore, there is a subgeometry partition in $\mathrm{PG}(3,q^2)$ with q^2+1 $\mathrm{PG}(1,q^2)$'s and $q^2(q-1)$ $\mathrm{PG}(3,q)$'s which geometrically lifts to the spread algebraically lifted from the spread in question in $\mathrm{PG}(3,q)$.

4 Subgeometry characterization of algebraically lifted spreads

In this section, we ask what sort of subgeometry mixed partitions of $\mathrm{PG}(3,q^2)$ ensure that the associated translation plane with spread in $\mathrm{PG}(7,q)$ is actually an algebraically lifted spread in $\mathrm{PG}(3,q^2)$. The idea of the following analysis is to consider the groups on the translation plane and ask what sort of actions the groups have on the mixed partition. Then we reverse the perspective and ask if groups acting on the mixed subgeometry partition induce the sort of groups that can characterize an algebraically lifted plane. Since we arrive back into a spread of $\mathrm{PG}(7,q)$, we looking for conditions to ensure the existence of an elation group E of order q^2, normalized by a Baer group B of order $q+1$, and both of these groups normalizing a group of order q^2-1, which will act as a kernel homology group.

Let $G=\mathcal{K}^*$, the multiplicative group of the field, and assume that we have a mixed subgeometry partition in $\mathrm{PG}(3,q^2)$ of q^2+1 $\mathrm{PG}(1,q^2)$'s and $q^2(q-1)$ $\mathrm{PG}(3,q)$'s. The group G of order q^2-1 fixes exactly q^2+1 components since the elements of the group are products of elements from a Baer group and a kernel homology group. This group has $q^2(q-1)$ orbits of length $q+1$, each of which is a regulus of $q+1$ components. The components of the Baer subplane π_0 become the projective subspaces isomorphic to $\mathrm{PG}(1,q^2)$. Furthermore, the reguli of $q+1$ components become the projective subspaces $\mathrm{PG}(3,q)$ and the orbits under G are the non-zero points of affine subspaces of order q, and these become points of the associated $\mathrm{PG}(3,q)$. So, G actually fixes each of the $\mathrm{PG}(1,q^2)$'s, and fixes each $\mathrm{PG}(3,q)$ pointwise.

We have another group H of order $q+1$ that permutes the mixed partition. We claim that H acts semi-regularly on the set of $(q^4-1)/(q-1)$ points of each $\mathrm{PG}(3,q)$. To see this, we note the regulus nets of degree $q+1$ of the translation plane correspond to the $\mathrm{PG}(3,q)$ and the group G fixes each subplane of order q and acts transitively on the non-zero points. If some element h fixes one of these subplanes of order q, then in the group $\langle h,G\rangle$ there is an non-identity element h^* which fixes a non-zero point. But this group also fixes each component of a net of degree q^2+1, which means that $h^*=1$. Hence, H is fixed-point-free on the $\mathrm{PG}(3,q)$'s.

However, H also fixes all $\mathrm{PG}(1,q^2)$'s, but fixes two points on each and has orbits of length $q+1$ on the remaining points q^2-1 points of each $\mathrm{PG}(1,q^2)$, since on a fixed component these points correspond to the intersection of Baer subplanes with the component.

If we think of the situation from the perspective of the mixed partition, we have a group H of order $q+1$ (or say of order divisible by $q+1$), and since G

ends up acting trivially on the mixed subgeometry partition, H will normalize the preimage G acting on the associated translation plane. That is, there will be a group G of order q^2-1 and a group of H has order divisible by $q+1$ acting on the translation plane, such that $G \cup \{0\}$ is a field $\mathcal{K}$ normalized by H. Furthermore, H will fix two components L and M.

If we isolate on a given fixed component L, then H and G, and hence GH, will fix exactly two subspaces over $\mathrm{GF}(q)$ of q^2 points — i.e. two 2-dimensional $\mathrm{GF}(q)$-subspaces on L. Furthermore, on these two fixed subspaces, it can only be that G acts transitively on the non-zero points. Hence, there is a group of order $q+1$ which fixes q^2+1 components and fixes a non-zero point. Hence, there is a Baer group of order $q+1$.

We register this property:

15 Lemma. *If we have a mixed partition of* $\mathrm{PG}(3,q^2)$ *of* q^2+1 $\mathrm{PG}(1,q^2)$*'s, each of which is fixed by a group* H *of order divisible by* $q+1$*, then in the geometrically lifted spread in* $\mathrm{PG}(7,q)$*, there will be a Baer group* B *of order* $(q+1)$.

A Sylow 2-subgroup of H permutes $q^2(q-1)$ $\mathrm{PG}(3,q)$'s and has orbits of length $q+1$. If H fixes one $\mathrm{PG}(3,q)$, then GH fixes a regulus in the translation plane, and since G acts transitively on the lines of the regulus, so does any Baer group of order $(q+1)$. In any case, we have a group of order $(q+1)$ fixing a line of the regulus. In the group GH, we note that the order of GH/G is exactly $q+1$, and GH has order $(q^2-1)(q+1)w/i$, where i is the order of the intersection. But then $w/i=1$. Thus, the order of GH is $(q^2-1)(q+1)$ and G is normal in GH. Hence, if H fixes one $\mathrm{PG}(3,q)$ there is a group of order $q+1$ $(GH)_L$, of order (q^2-1) which fixes L. Therefore, $(GH)_L$ contains the kernel $\mathrm{GF}(q)$-homology group. G is a cyclic group of order (q^2-1), which is normalized by H. Therefore, H acting on G as an automorphism group is Abelian. We note that $H/H\cap G$ has order $(q+1)$, and must be Abelian. Note also that it follows that $H\cap G$ is contained in the kernel homology group of order $q-1$. So, $(GH)_L$ has order q^2-1 and fixes L. If H fixes all $\mathrm{PG}(3,q)$'s then there are q^4-q^2 subgroups $(GH)_M$. In GH, how many groups of order q^2-1 are there, which contain the kernel $q-1$ group? GH/K^* has order $(q+1)^2$, and contains a cyclic group of order $q+1$.

We also have an elation group E of order q^2 acting on the translation plane of order q^4, which has q^2 orbits of length q^2. Our original Baer group B of order $q+1$ has a fixed-point space which is a Baer subplane π_0, which shares the axis of E and has exactly one component in each E-orbit of components. Since our field $\mathcal{K}$ of order q^2 is formed using the existence of the kernel group of order q^2-1 and the Baer group, we see that under E, we have exactly q^2 conjugate fields $\mathcal{K}_u$, for $u\in\mathrm{GF}(q^2)$, that mutually share exactly the kernel homology field GF(q). Since these groups will have the same orbit sizes, we then obtain a set of q^2 mixed subgeometry partitions of PG$(3, q^2)$, each of which admits the kernel homology group acting permutation-isomorphically. Hence we obtain the following theorem.

16 Theorem. *Let π be a translation plane of order q^4 that has been algebraically lifted from a plane with spread in* PG$(3, q)$.

(1) Then there are q^2 groups G_i of order $q^2 - 1$ that construct by retraction a set of q^2 mixed subgeometry partitions $\mathcal{M}_i$, $i = 1, 2, \ldots,$ all with $q^2 + 1$ PG$(1, q^2)$*'s and with $q^2(q-1)$* PG$(3, q)$*'s.*

(2) These mixed partitions pairwise mutually share precisely one common PG$(1, q^2)$.

(3) Each of these subgeometry partitions admits a collineation group H_i of order $q+1$ that permutes the mixed partition, which fixes each PG$(3, q)$ *and has orbits of length $q+1$ on the $(q^4-1)/(q-1)$ points of each* PG$(3, q)$.

(4) H_i also fixes all PG$(1, q^2)$*'s, fixes two points and has orbits of length $q+1$ on the remaining points of each* PG$(1, q^2)$.

We now wish to characterize our algebraically lifted spreads using mixed subgeometry partitions.

17 Definition. Assume that PG$(3, q^2)$ admits q^2 mixed subgeometry partitions $\mathcal{M}_i$, $i = 1, 2, \ldots, q^2$, all with q^2+1 PG$(1, q^2)$'s and with $q^2(q-1)$ PG$(3, q)$'s that pairwise share precisely one common PG$(1, q^2)$ and no common PG$(3, q)$'s. Furthermore, assume that each of these subgeometry partitions admits a collineation group H_i of order $q+1$, which fixes each PG$(3, q)$ and has orbits of length $q+1$ on the $(q^4-1)/(q-1)$ points of each PG$(3, q)$. Also assume that H_i fixes all PG$(1, q^2)$'s corresponding to that mixed partition, fixes two points of each, and has orbits of length $q+1$ on the remaining points of each PG$(1, q^2)$.

If the set of mixed subgeometry partitions $\mathcal{M}_i$, for $i = 1, 2, \ldots, q^2$, all geometrically lift to the same translation plane π of order q^4, then π shall be said to be a q^2-fold geometric lift.

Our main theorem of q^2-fold geometric lifts is the following.

18 Theorem. *If π is a q^2-fold geometric lift then π is a translation plane which has been algebraically lifted from a translation plane with spread in* PG$(3, q)$.

We give the proof as a series of lemmas. To use Theorem 7, we need to establish three things: (1) There is a Baer group B of order $q+1$, (2) there is a kernel homology group C of order q^2-1, (3) there is an elation group E of order at least q^2 such that $[E, B] \neq 1$.

19 Lemma. *Each mixed subgeometry partition S_i geometrically lifts to π, which then admits a collineation group G_i of order q^2-1, which contains the subkernel homology group of order $q-1$. Then $G_i \cap G_j = \mathrm{GF}(q)^*$.*

Proof. Since the subgeometry partitions are distinct, then it follows from Johnson and Mellinger [6] that for all $i \neq j$, $G_i \cap G_j = \mathrm{GF}(q)^*$, the kernel homology group of order $q-1$. $\boxed{QED}$

20 Lemma. *The group H_i acting on the spread π normalizes the group G_i, for $i = 1, 2, \ldots, q^2$.*

Proof. This is true due to the retraction construction. $\boxed{QED}$

21 Lemma. *G_iH_i contains a Baer group B_i of order $(q+1)$.*

Proof. Apply Lemma 15. $\boxed{QED}$

22 Lemma. *The Baer groups B_i and B_j, for $i \neq j$, are disjoint.*

Proof. We know that all groups fix a common component L, which produces under retraction the unique fixed $\mathrm{PG}(1, q^2)$ of the set of mixed subgeometry partitions. Let the Baer subplanes pointwise fixed by B_k be denoted by π_k. Assume that g is in $B_i \cap B_j$. The common fixed component L intersects π_k in a 2-dimensional subspace X_k. If g is not 1 then g fixes $\pi_i \cap L$ and $\pi_j \cap L$ pointwise, a contradiction unless $\pi_i \cap L = \pi_j \cap L$, and this in turn means that $\pi_i = \pi_j$. Now in our previous analysis we noted that each G_iH_i will fix exactly two 'points' of L and these two points correspond to the Baer subplane fixed by B_i and to its unique 'coaxis' Baer subplane (that Baer subplane on the same net as the Baer subplane, which is also fixed by the Baer group).

But this, of course, means that mixed subgeometry partitions corresponding to G_i and G_j have their sets of common $\mathrm{PG}(1, q^2)$'s equal, which is contrary to our assumptions. $\boxed{QED}$

23 Lemma. *The union of the components of the Baer subplanes π_i, for $i = 1, 2, \ldots, q^2$, together with L is the spread for π.*

Proof. From the previous lemma, all of the Baer groups are necessarily disjoint and each Baer subplane π_i shares exactly L with π_j, for $i \neq j$. Consider the group $G = \left\langle G_i, H_i; i = 1, 2, \ldots, q^2 \right\rangle$. Since there are at least q^2 Baer groups, each with q^2+1 components that mutually share exactly one component L, within a translation plane of order q^4, it follows that the union of the set of components of the Baer subplanes forms a disjoint cover of the components not equal to L. $\boxed{QED}$

24 Lemma. *Let W be the group of central collineations with axis L in the group $\left\langle \cup_{i=1}^{q^2} G_i \right\rangle$.*

(1) Then W has order at least q^2.

(2) $G_iB_i = G_iH_i$.

(3) B_i *is a Baer group of order* $q+1$ *and the component orbits of* B_i *are the same as the component orbits of* G_i.

Proof. Since G_i and G_j share L, it is clear that each has the same orbit of length q^2-1 on L. That is, $\langle G_i, G_j\rangle$ fixes each orbit of length q^2-1 on L. This means that given any element g_i in G_i, there is an element g_j in G_j whose action on L is identical. Hence, $g_ig_j^{-1}$ acts identically on L, that is, $g_ig_j^{-1}$ is an elation with axis L. Assume that for $j \neq k$, $g_ig_j^{-1} = g_ig_k^{-1}$. Then $g_j = g_k$ in $G_j \cap G_k$. We know that this intersection is the kernel homology group K^*, of order $q-1$. This means for each element g_i of G_i of order dividing q^2-1 but not $q-1$, there is a central collineation $g_ig_j^{-1}$, where g_j also has order dividing q^2-1 but not $q-1$ and no two g_j and g_k are equal in $g_ig_j^{-1}$ or $g_ig_k^{-1}$. Hence, we have a generated central collineation group W with axis L of order at least q^2. This proves part (1).

Since G_iH_i has order $(q^2-1)(q+1)$ and contains a Baer group B_i of order $q+1$, then $G_iH_i = G_iB_i$. Since G_i and H_i fix the same $\mathrm{PG}(1,q^2)$'s, it follows immediately that there is a unique Baer group in G_iB_i, which is then normalized by G_i. Furthermore, our assumptions then show that the component orbits of B_i are the same as those of G_i; this proves (2) and (3). $\boxed{QED}$

25 Lemma. *Let* M *be any component of any orbit of* G_iB_i *of length* $q+1$. *Then there is a cyclic subgroup* $C_{i,M} = (G_iB_i)_M$ *of order* q^2-1. *Moreover,* $C_{i,M} \cup \{0\}$ *is a field. Furthermore,* $G_iC_{i,M} = G_iB_i$.

Proof. Take a component M of any orbit of length $q+1$. Then there is a subgroup $(G_iB_i)_M$ of G_iB_i of order q^2-1, which fixes M and must act fixed-point-free on M, since this group will also fix each of the q^2+1 components fixed by G_i. Recall that both G_i and B_i are cyclic of orders q^2-1 and $q+1$ respectively and that G_iH_i fixes two 2-dimensional $\mathrm{GF}(q)$-subspaces on L, and B_i fixes one of these pointwise. Hence, the action of $(G_iB_i)_M$ on that particular subspace is faithful and equal to the action of G_i on that subspace. Hence, it follows that this group $(G_iB_i)_M$ is cyclic of order q^2-1, from which it follows that $C_{i,M} \cup \{0\}$ is a field (see Johnson and Mellinger [6]). We note that in G_i, we have the kernel subgroup of order $q-1$ and G_i has an orbit of length $q+1$ containing M, hence, $G_iC_{i,M} = G_iB_i$. $\boxed{QED}$

26 Lemma. G_iH_i *is faithful on* L *and* $C_{i,M}$ *commutes with* G_i.

Proof. To see this, just recall that G_iH_i fixes q^2+1 components. Since $G_i \cup \{0\}$ is a field, of order q^2, it follows that G_iH_i is in $GL(2,q^2)$ acting on L, with G_i acting as the kernel homology group of an ambient Desarguesian plane of order q^2 in L. Hence, H_i actually must centralize G_i as $G_iH_i = G_iB_i$, forcing it to be in $GL(2,q^2)$, as opposed to simply $\Gamma L(2,q^2)$ (as B_i then becomes an affine homology group of the associated Desarguesian affine plane). Hence, $C_{i,M}$ commutes with G_i. $\boxed{QED}$

27 Lemma. *$G_iH_i \cup \{0\}$ contains exactly two fields*

$$K_\alpha = \left\{ \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix} ; \alpha \in \mathrm{GF}(q^2) \right\},$$

$$K_{\alpha^q} = \left\{ \begin{bmatrix} \alpha^q & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha^q & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix} ; \alpha \in \mathrm{GF}(q^2) \right\}.$$

Proof. Since $C_{i,M}$ commutes with G_i, it follows that $C_{i,M}$ fixes all components in the G_i-orbit of M. On L, choose coordinates so that if $L = \{(x_1, y_1)\}$, then $x_1 = 0$ and $y_1 = 0$ define the intersections of L with π_i and π_i^*, the Baer subplane pointwise fixed by B_i and the coaxis subplane fixed by B_i. We know that $C_{i,M}$ acts like G_i on π_i and leaves the other invariant. Assume that $y_1 = 0$ is $L \cap \pi_i$. Let $G_i \cup \{0\}$ acting on $y_1 = 0$ be denoted by F_{y_1}, and acting on $x_1 = 0$ let the action be denoted by F_{x_1}, so that both F_{y_1} and F_{x_i} are fields isomorphic to $\mathrm{GF}(q^2)$. Then we may assume that G_i has the form

$$\left\langle \begin{bmatrix} \alpha & 0 \\ 0 & f(\alpha) \end{bmatrix} ; \alpha \in F_{y_1} \right\rangle$$

for some function $f\colon F_{y_1} \to F_{x_1}$, acting on L. Since $G_i \cup \{0\}$ is a field, f is an isomorphism of fields. If we identify the fields F_{y_1} and F_{x_1}, then $f(\alpha) = \alpha^\sigma$, where σ is an automorphism of F_{y_1}. Furthermore, since G_i contains the $\mathrm{GF}(q)$-scalar mappings, it follows that $\sigma = q$ or q^2. Since $C_{i,M}$ then must have the same individual group on each of these two subspaces as does G_i, and since $C_{i,M} \cup \{0\}$ is a field, we see that

$$\left\langle \begin{bmatrix} \alpha^\beta & 0 \\ 0 & \alpha^\delta \end{bmatrix} ; \alpha \in F_{y_1} \right\rangle, \beta, \delta \text{ automorphisms of } F_{y_1}.$$

Now the group $G_iC_{i,M} = G_iB_i$, so consider

$$\left\langle \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^\sigma \end{bmatrix}, \begin{bmatrix} \gamma^\beta & 0 \\ 0 & \gamma^\delta \end{bmatrix} ; \alpha, \gamma \in F_{y_1} - \{0\} \right\rangle.$$

In order to create a Baer group of order $q+1$, within $G_iC_{i,M}$, then the action on $y_1 = 0$ is forced to be identical in both G_i and $C_{i,M}$. Hence, we may assume that $\beta = 1$. If we choose a basis so the fixed components under $G_iB_i = G_iC_{i,M}$ include $x = 0, y = 0, y = x$, then, we may assume that $G_i \cup \{0\}$ has the diagonal form

$$\left\langle Diag \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{\sigma} \end{bmatrix}; \alpha \in F_{y_1} \right\rangle, \sigma \text{ an automorphism of } F_{y_1}, \text{ equal to } q \text{ or } 1$$

and $C_{i,M} \cup \{0\}$ has the diagonal form

$$\left\langle Diag \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{\delta} \end{bmatrix}; \alpha \in F_{y_1} \right\rangle, \delta \text{ an automorphisms of } F_{y_1},$$

where $\alpha^{\sigma} = \alpha^{\delta}$ if and only if α is GF(q) (the subfield of F_{y_1} of order q), as the groups share exactly the kernel homologies of order $q - 1$. Hence, the action of the Baer group B_i on L is

$$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & \alpha^{\sigma-\delta} \end{bmatrix}; \alpha \in F_{y_1} - \{0\} \right\rangle.$$

But the Baer group is cyclic of order $q + 1$, which means that for α in GF(q), we have $\alpha^{\sigma-\delta} = 1$. As we have two distinct fields, we may assume that $\{1, q\} = \{\sigma, \delta\}$. Hence, we have three possible fields.

$$\begin{aligned}
K_{\alpha} &= \left\{ \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}; \alpha \in \mathrm{GF}(q^2) \right\}, \\
K_{\alpha^q} &= \left\{ \begin{bmatrix} \alpha^q & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha^q & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}; \alpha \in \mathrm{GF}(q^2) \right\}, \\
K^{\alpha^q} &= \left\{ \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha^q & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha^q \end{bmatrix}; \alpha \in \mathrm{GF}(q^2) \right\}.
\end{aligned}$$

However, taking $\alpha = \beta^q$ in the last field, we see that $K_{\alpha^q} = K^{\alpha^q}$. This completes the proof of the lemma. $\boxed{QED}$

28 Lemma. *$C = C_{i,M} = C_{i,N}$, for all components M and N. Hence,*

(1) C is a kernel homology subgroup of order $q^2 - 1$, and

(2) there are two Baer groups of order $q + 1$ in CB_i, B_i and B_i^* where B_i and B_i^* share the same components.

PROOF. By the previous lemma, there are exactly two subfields in G_iB_i and G_i is one of them. Hence, $C_{i,M} \cup \{0\} = C_{i,N} \cup \{0\}$, and this implies that C is a kernel homology subgroup of order $q^2 - 1$. This proves (1). Since we now have a kernel homology group, then noting that each Baer group B_i fixes two Baer subplanes of a given net fixed componentwise, it follows that by appropriate multiplication by a kernel homology, we may create an associated Baer group of order $q + 1$ with axis the second Baer subplane fixed by B_i, the first being the axis of B_i. $\boxed{QED}$

This means that the kernel of π contains $\mathrm{GF}(q^2)$. Thus, $G_jH_j = G_jC$, for all $i = 1, 2, \ldots, q^2$. In order that we have an algebraically lifted plane in π, using Corollary 6, it suffices to show that there is an elation group E of order at least q^2 such that $[E, B_1] \neq 1$. We know that we have an elation group E of order at least q^2, which is normalized by all Baer groups B_i of order $q + 1$.

Since we take C as the kernel homology group, we may assume that C has the form K_a and G_1 has the form K_{a^q}, where we are assume that $x = 0, y = 0, y = x$ are fixed by G_1. Since in this group lies a Baer group of order $q+1$, we see that B_1 has the following form:

$$B_1 = \left\langle \begin{bmatrix} a^{q-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a^{q-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; a \in F_{y_1} \right\rangle .$$

At this point, we use the notation $\mathrm{GF}(q^2)$ to denote F_{y_1}.

In this form, the elation subgroup E' of the full elation group E that commutes with B_1 has the following form:

$$\left\langle \begin{bmatrix} I & R \\ 0 & I \end{bmatrix} ; R = \begin{bmatrix} T(u) & 0 \\ 0 & u \end{bmatrix} ; u \in \lambda \subseteq \mathrm{GF}(q^2) \right\rangle$$

(we are not yet claiming that the central collineation group W, defined in Lemma 24, is an elation group).

29 Lemma. *Let the nets componentwise fixed by B_i be denoted by N_{B_i}. Then $B_1^W = \{B_i; i = 1, 2, \ldots, q^2\}$. That is, W is transitive on the set of q^2 Baer groups B_i, for $i = 1, 2, \ldots, q^2$.*

W contains an elation subgroup E that is transitive on the set of q^2 Baer groups B_i.

PROOF. Suppose that B is a Baer group of order $q + 1$ in G. Since G fixes exactly two 'points' on L, it follows by the previous lemma that we may assume that B and B_i fix the same points on L and fix the same coaxis on L; B and B_i have the same axis and coaxis on L. Assume without loss of generality that B

fixes the component $y=0$. If B also fixes another component fixed by B_1 then elements of B have the form:

$$\left\langle \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & a^{q-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle,$$

since it must fix a component $y = x \begin{bmatrix} M(u_0) & 0 \\ 0 & u_0 \end{bmatrix}$, for some $u \in \mathrm{GF}(q^2)$ (as these are the only components that can be fixed by B_1). Since the order of B is $q+1$, then c and d must both have orders dividing $q+1$ and clearly one of these elements has order equal to the order of a^{1-q}. If the set of components fixed by B_1 is

$$\left\{ y = x \begin{bmatrix} M(u) & 0 \\ 0 & u \end{bmatrix}; u \in \mathrm{GF}(q^2) \right\},$$

then

$$\begin{bmatrix} c & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & a^{q-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

maps $y = x \begin{bmatrix} M(u) & 0 \\ 0 & u \end{bmatrix}$ onto

$$y = x \begin{bmatrix} M(u)c^{-1}a^{q-1} & 0 \\ 0 & ud^{-1} \end{bmatrix}$$

and so can fix one if and only if $ud^{-1} = u$. Hence, $d^{-1} = 1$, if B and B_1 fix at least two components other than L. If $c^{-1}a^{q-1}$ is not 1, then

$$\begin{bmatrix} M(u)c^{-1}a^{q-1} & 0 \\ 0 & u \end{bmatrix} - x \begin{bmatrix} M(u) & 0 \\ 0 & u \end{bmatrix}$$

is non-singular. Hence, $B = B_1$ if there are at least two common fixed components not equal to L. Therefore B shares exactly one fixed component with each of the B_i, for $i = 1, 2, \ldots, q^2$, other than L. If B fixed a point common to $\pi_i - L$ then B would be forced equal to B_i. We are in the dimension-2 situation, and on the net N_B of degree q^2+1, the fixed-point space and its coaxis space are $\mathrm{GF}(q^2)$-subspaces. Hence, on $y=0$, regarded as the Desarguesian affine plane of order q^2, we note that the groups B and B_1 induce, on $y=0$, affine homology groups of order $q+1$, and the axes on $y=0$ cannot be the same and hence are disjoint (on $y=0$).

Now we know that W is a central collineation group of order at least q^2 that has a subset of at least $q^2 - 1$ elements that cannot commute with B_1. This means that none of these elements will map a component of N_{B_1} not equal to $x = 0$ back into N_{B_1}, for, if so, then the Baer subplane π_1 would be fixed and hence so would N_{B_1} and thus also the element would centralize B_1. Hence, W maps N_{B_1} into at least q^2-1 other nets of degree q^2+1 that are mutually disjoint except for $x = 0$. Consider one of these nets $N_{B_1^w}$. So, the q^2 other components not equal to L must be distributed one each to the remaining $q^2 - 1$ nets N_{B_i}, a contradiction. Hence, $N_{B_1^w} = N_{B_i}$.

Assume that W is not an elation group. Then some element w of W is an affine homology with axis L and coaxis M. But we have noted that the union of the nets N_{B_i} is the spread for π, and hence M is in an orbit of length at least q^2. Hence, by André's theorem, there is an elation group of this order as well with the same orbit. $\boxed{QED}$

So, we have a Baer group B of order $q + 1$, a kernel homology group C of order $q^2 - 1$, and an elation group E of order at least q^2 such that $[E, B] \neq 1$. Hence, we see that the plane π is an algebraically lifted plane. This completes the proof of the theorem.

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